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ACTIONS OF CYLINDERS

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Let $\mathscr{C}$ be a foliation without first integrals given by a C^r, $r\geqslant 2$, nonsingular action of the cylinder $T^k\times\mathbb{R}^l$, $k\geqslant 1$, on a closed, orientable, connected m-manifold M and $A^r(\mathbb{R}^p, M, \mathscr{C})$ be the space of all C^r nonsingular actions of $\mathbb{R}^p$, $p=k+l$, on M with underlying foliation $\mathscr{C}$. To study this space we consider three basic equivalence relations: *linear equivalence*, *conjugation* and *concordance*. We first show that any F in $A^r(\mathbb{R}^p, M, \mathscr{C})$ is, up to linear equivalence, an action of the cylinder $T^k\times\mathbb{R}^l$. When $k=m-2$, $l=1$ we show the degeneracy of the characteristic mapping α_F of actions F in $A^r(\mathbb{R}^p, M, \mathscr{C})$. Finally we show that, up to conjugation and linear equivalence, any two C^r nonsingular actions of the cylinder $T^{m-2}\times\mathbb{R}$ on M are always concordant.

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Introduction

This paper is a continuation of our paper "Actions of $\mathbb{R}^p$ on closed manifolds", [1]. Here we consider nonsingular actions of the cylinder $T^k\times\mathbb{R}^l$ on a closed orientable connected m-manifold M. Let $\mathscr{C}$ be a foliation given by a C^r, $r\geqslant 2$, nonsingular action of the cylinder $C:(T^k\times\mathbb{R}^l)\times M\to M$ and $A^r(\mathbb{R}^p, M, \mathscr{C})$, $p=k+l$, be the space of all nonsingular C^r actions F of $\mathbb{R}^p$ on M with underlying foliation $\mathscr{C}$. To study this space we consider three basic equivalence relations: *linear equivalence*, *conjugation* and *concordance*.

Two actions F and G of $\mathbb{R}^p$ on M are said to be *linearly equivalent* if one is a linear reparametrization of the other. They are *conjugate* if there is a diffeomorphism $\phi:M\to M$ such that $F(v,\phi(x))=\phi G(v,x)$ for all $v\in\mathbb{R}^p$ and $x\in M$. First we prove a reparametrization lemma.

1.1. Lemma. *Let K and L be nonsingular left C^r actions, $r\geqslant 2$, of a connected and simply connected Lie group G on M. If K and L have the same underlying foliation, then there exists a unique C^r mapping $\rho: G\times M\to M$ with the following properties*:

(i) *$\rho(e,x)=e$ for each $x\in M$, e being the identity of G.*

(ii) *The mappings $\rho_x: G\to G$, $\rho_x(g)=\rho(g,x)$ are C^r diffeomorphisms.*

(iii) *$K(g,x)=L(\rho(g,x),x)$ for every g in G and x in M.*

A function $f: M \to \mathbb{R}$ is a *first integral* of a foliation $\mathscr{F}$ if it is nonconstant but constant along the leaves of $\mathscr{F}$. In Section 2 we study the space $A^r(\mathbb{R}^p, M, \mathscr{C})$ under the point of view of linear equivalence.

2.1. Theorem. *If $\mathscr{C}$ has no C^r first integrals, then each action F in $A^r(\mathbb{R}^p, M, \mathscr{C})$ is linearly equivalent to an action of the cylinder $T^k \times \mathbb{R}^l$.*

In [1] we associate to a nonsingular C^r, $r \geq 2$, action F of $\mathbb{R}^p$ on M a symmetric $(q+1)$-linear mapping α_F of $\mathbb{R}^p$ into $H^{2q+1}_{\mathrm{deR}}(M)$, $q = m - p$, called the *characteristic mapping* of F. Here we prove:

2.2. Corollary. *If $\mathscr{C}$ is the underlying foliation of a nonsingular action of the cylinder $C : (T^{m-2} \times \mathbb{R}) \times M \to M$ and $\mathscr{C}$ has no C^r first integrals, then the characteristic mapping α_F is a degenerate bilinear mapping for every F in $A^r(\mathbb{R}^{m-1}, M, \mathscr{C})$. If $M = T^3$, then actually $\alpha_F = 0$.*

In Example 2.3, we use Corollary 2.2, to show the existence of a foliation $\mathscr{F}$, with a finite number of compact orbits (tori), all the other being cylinders, which is not the underlying foliation of an action of the cylinder.

We show in Example 2.5 that Theorem 2.1 is false if $\mathscr{C}$ has C^r first integrals. In Remark 3.13 we show that Corollary 2.2 is false if $\mathscr{C}$ is given by an action C of $T^k \times \mathbb{R}^l$ on M where $0 < k < m-2$.

We proved in [1, 1.12] that the set of nonsingular C^r actions $F: \mathbb{R}^2 \times M^3 \to M^3$ with a nondegenerate characteristic mapping α_F is open in the C^1-topology. We show in Example 2.6 the existence of an open set V of C^∞ nonsingular actions of $\mathbb{R}^2$ on T^3 such that every action F in V has a vanishing characteristic mapping, answering negatively [1, 1.13].

Two nonsingular actions of the cylinder $T^{m-2} \times \mathbb{R}$ on M are not in general conjugate, even if they have the same underlying foliation (see Example 3.1). For this reason, we consider in Section 3 another equivalence relation called *concordance* of actions. It has a close relation with the notion of maximal class of 1-forms [4, VI] and with the foliated cohomology, introduced by Reinhart in [5]. This cohomology appears naturally in the study of actions [2]. Here we prove:

3.9. Theorem. *Let $\mathscr{C}$ be the underlying foliation of a nonsingular C^r, $r \geq 2$, action C of the cylinder $T^{m-2} \times \mathbb{R}$ on M. If $\mathscr{C}$ has no C^r first integrals, then every action in $A^r(\mathbb{R}^{m-1}, M, \mathscr{C})$ is concordant to a linear reparametrization of C.*

In Theorem 3.11 we show that, up to conjugation and linear equivalence, any two C^r nonsingular actions of the cylinder $T^{m-2} \times \mathbb{R}$ on M are always concordant. In Remark 3.13 we show that there are many classes of concordance of actions of the cylinder $T^k \times \mathbb{R}^l$ on M if $0 < k < m-2$.

1. Actions of Lie groups

Let M be a closed orientable connected m-manifold and G a connected and simply connected Lie group. If $F: G\times M\to M$ is a nonsingular differentiable left action, the foliation on M given by the orbits of F will be referred to as the *underlying foliation* of F. Let $\mathcal{B}=\{v_1,\ldots,v_n\}$ be a basis of the Lie algebra of G. The *infinitesimal generators* of F with respect to $\mathcal{B}$ are the vector fields V_j on M given by

$$V_j(x)=\mathrm{D}F_x(0)\cdot v_j;\quad j=1,\ldots,n,$$

where $F_x: G\to M$, $F_x(g)=F(g,x)$, $x\in M$ and $g\in G$. The action F is said to be of class C^r, $r\geqslant 2$, if its infinitesimal generators are of class C^r. Since F is nonsingular, then $V=\{V_1,\ldots,V_n\}$ is a frame of M.

In this section we prove a lemma on reparametrization of actions.

1.1. Lemma. *Let K and L be nonsingular left C^r actions, $r\geqslant 2$, of a connected and simply connected Lie group G on a closed orientable connected m-manifold M. If K and L have the same underlying foliation $\mathcal{F}$, then there exists a unique C^r mapping $\rho: G\times M\to G$ with the following properties*:

(i) *$\rho(e,x)=e$ for each x in M, e being the identity element of G.*

(ii) *The mappings $\rho_x: G\to G$ given by $\rho_x(g)=\rho(g,x)$ are C^r diffeomorphisms for each x in M.*

(iii) *$K(g,x)=L(\rho(g,x),x)$ for each $g\in G$ and x in M.*

Proof. Since K and L have both the same underlying foliation $\mathcal{F}$, then for each x in M there exists a unique C^r diffeomorphism

$$\boldsymbol{\rho}_x: G/K(x)\to G/L(x),$$

such that $\boldsymbol{K}_x=\boldsymbol{L}_x\circ\boldsymbol{\rho}_x$ where $K(x)$ and $L(x)$ are the isotropy groups at x of K and L, respectively and $\boldsymbol{K}_x$, $\boldsymbol{L}_x$ are the quotient mappings induced by K_x and L_x on the homogeneous spaces $G/K(x)$ and $G/L(x)$. We note that since K and L are nonsingular actions, then $K(x)$ and $L(x)$ are discrete subgroups of G and the above homogeneous spaces have both the same dimension as G. Since G is simply connected, then there exists a unique lifting $\rho_x: G\to G$ of $\boldsymbol{\rho}_x$ such that $\rho_x(e)=e$. Moreover ρ_x is a C^r diffeomorphism and satisfies (iii). To show that ρ is a C^r mapping, let $X=\{X_1,\ldots,X_n\}$ and $Y=\{Y_1,\ldots,Y_n\}$ be the infinitesimal generators of K and L with respect to a basis $\mathcal{B}=\{v_1,\ldots,v_n\}$ of the Lie algebra of G, respectively. Derivation of both sides of (iii) leads to the equation

$$\mathrm{D}K_x(g)=\mathrm{D}L_x(\rho_x(g),x)\circ\mathrm{D}\rho_x(g)$$

or in matrix notation

$$Y(K(g,x))=A(g,x)X(K(g,x)), \tag{1}$$

where $A(g, x)$ is the transpose of the matrix of $D\rho_x(g)$ in the basis $\mathcal{B}$. If σ is a C^∞ curve in G such that $\sigma(0)=e$ and $\sigma(1)=g$, then assuming that G is embedded in a Euclidean space, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(\sigma(t), x)] = A(\sigma(t), x)\cdot b(t), \tag{2}$$

where $b(t)$ is the matrix of $\sigma'(t)$ in the basis $\mathcal{B}$. Integration of (2) gives

$$\rho(g, x) = e + \int_0^1 A(\sigma(t), x)\cdot b(t)\,\mathrm{d}t. \tag{3}$$

It follows from (1) that $A: G\times M \to \mathrm{GL}(n; \mathbb{R})$ is a C^r mapping. Thus from (ii) and (3) we see that ρ is C^r. $\square$

The group Aut G of all automorphisms of G acts naturally on the left on the space $A^r(G, M)$ of all nonsingular C^r left actions of G on M by $(A, F)\to F_A$ where $F_A(g, x) = F(A(g), x)$. For this reason we say that two actions K and L in $A^r(G, M)$ are *linearly equivalent* if $L = K_A$ for some automorphism A of G. We denote by $A^r(G, M)$ the set of all classes under this equivalence relation.

2. Actions of the cylinder and first integrals

In this section we study the space $A^r(\mathbb{R}^{k+l}, M, \mathscr{C})$ of all C^r nonsingular actions of $\mathbb{R}^{k+l}$ on M whose underlying foliation $\mathscr{C}$ is given by an action C of the cylinder $T^k\times\mathbb{R}^l$.

A function $f: M\to\mathbb{R}^q$ which is constant on each leaf of a foliation $\mathscr{F}$ of M is called a *first integral* if it is nonconstant. Existence of differentiable first integrals for a codimension-q foliation implies, by a Sard's theorem, the existence of compact leaves.

Let C_0 be an action of $T^k\times\mathbb{R}^l$ on M, $p=k+l$, and $\exp:\mathbb{R}^p\to T^k\times\mathbb{R}^l$ given by $\exp(u, v) = (e^{2\pi i u_1}, \ldots, e^{2\pi i u_k}, v_1, \ldots, v_e)$. The action of $\mathbb{R}^p$ on M given by $C(u, v, x) = C_0(\exp(u, v), x)$ is called the *canonical lifting* of C_0 [1, 2].

2.1. Theorem. *Let $\mathscr{C}$ be the underlying foliation of a C^r, $r\geq 2$, nonsingular action C_0 of the cylinder $T^k\times\mathbb{R}^l$, $k\geq 1$, on a closed connected orientable m-manifold M. If $\mathscr{C}$ has no C^r first integrals, then each action F in $A^r(\mathbb{R}^{k+l}, M, \mathscr{C})$, $r\geq 2$, is linearly equivalent to the canonical lifting of an action of the cylinder $T^k\times\mathbb{R}^l$.*

Proof. By Lemma 1.1 there exists a C^r mapping $\rho:\mathbb{R}^{k+l}\times M\to\mathbb{R}^{k+l}$ such that $F(t, x) = C(\rho(t, x), x)$.

For each j, $1\leq j\leq p$, $p=k+l$, we have a unique function $\tau_j: M\to\mathbb{R}^p$ such that $\rho(\tau_j(x), x) = e_j$ for every x in M, $\{e_1, \ldots, e_p\}$ being the standard basis of $\mathbb{R}^p$. By the implicit function theorem τ_j is C^r. Since $\tau_j(F(t, x))$ belongs to the isotropy group of F at x it follows that τ_j is constant on the leaves of $\mathscr{C}$ and thus by hypothesis a constant function. Let A be the $(k+l)\times(k+l)$ matrix whose columns are $A^j = \tau_j$ for $1 \leq j \leq k$ and $A^j = e_j$, $k+1 \leq j \leq k+l$. It is easy to see that $F_A(v, x) = F(A\cdot v, x)$ is the canonical lifting of an action of the cylinder, proving the theorem. □

We now recall the definition of the characteristic mapping of a nonsingular action $F:\mathbb{R}^p \times M \to M$ as defined in [1]. Let $X = \{X_1, \ldots, X_p\}$ be the p-frame of the infinitesimal generators of F and $\xi = \{\xi_1, \ldots, \xi_p\}$ a p-coframe adapted to F, i.e., $\xi_i(X_j) = \delta_{ij}$. The *characteristic mapping* α_F of F is the $(q+1)$-linear mapping, $q = m-p$,

$$\alpha_F : \mathbb{R}^p \times \cdots \times \mathbb{R}^p \to H^{2q+1}_{\mathrm{de}\mathbb{R}}(M) \tag{4}$$

defined by $\alpha_F(e_{i_1}, \ldots, e_{i_p}) = \xi_{i_1} \wedge \mathrm{d}\xi_{i_2} \wedge \cdots \wedge \mathrm{d}\xi_{i_p}$. It is shown in [1, 1] that α_F does not depend on the adapted coframe taken. We proved in [1, 2.8] that if $C:(T^{m-2}\times \mathbb{R}) \times M \to M$ is a nonsingular C^r, $r \geq 2$, action of the cylinder, then its characteristic mapping is a degenerate bilinear mapping. Thus if $F:\mathbb{R}^{m-1} \times M \to M$ is an action linearly equivalent to C, then α_F is also degenerate [1, 1.3]. We have proved the following:

2.2. Corollary. *Let $\mathscr{C}$ be the underlying foliation of a nonsingular C^r, $r \geq 2$, action of the cylinder $C:(T^{m-2}\times\mathbb{R})\times M \to M$ on a closed connected orientable m-manifold M. If $\mathscr{C}$ has no C^r first integrals, then the characteristic mapping α_F is a degenerate bilinear mapping for every $F \in A^r(\mathbb{R}^{m-1}, M, \mathscr{C})$.*

We observe that if $M = T^3$ in the above corollary, then actually $\alpha_F = 0$. This follows from [1, 3.1] and Theorem 2.1. Sá Earp in [7] has shown that there are actions $F:\mathbb{R}^2 \times T^3 \to T^3$ with a finite number of compact orbits (all the other orbits being either cylinders or planes) with a nonvanishing characteristic mapping. Such actions are not linearly equivalent to actions of the cylinder $S^1 \times \mathbb{R}$.

2.3. Example. Let $\mathscr{F}$ be the underlying foliation of the action $E:\mathbb{R}^2 \times T^3 \to T^3$ given by the vector fields

$$E_1 = \frac{\partial}{\partial x} - a(z)\frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial y} - b(z)\frac{\partial}{\partial z},$$

where $a(z)$ has flat zeros at $z_0 = 0$, $z_1 = \frac{1}{3}$, $z_2 = \frac{2}{3}$ and $z_3 = 1$. The commutativity of E_1 and E_2 implies $b(z)/a(z) = c_i$, $z_i < z < z_{i+1}$, for $i = 0, 1, 2$. Thus in each tube $T^2 \times (z_i, z_{i+1})$ the foliation $\mathscr{F}$ is given by the closed 1-form $w_i = \mathrm{d}x + c_i\,\mathrm{d}y + (1/a(z))\,\mathrm{d}z$ and all the orbits are cylinders if the constant c_i is rational and planes otherwise. By [1, 4.5] a 1-form $\mu = f(z)\,\mathrm{d}x + g(z)\,\mathrm{d}y$ is $\mathscr{F}$-closed iff $E_1 g = E_2 f$. Thus $g(z) = c_i f(z) + d_i$ for $z_i \leq z \leq z_{i+1}$ where the constants d_i are the solutions of the linear system

$$\begin{aligned} d_1 \qquad - d_3 &= (c_3 - c_1) f(z_0), \\ d_1 - d_2 \qquad &= (c_2 - c_1) f(z_1), \\ d_2 - d_3 &= (c_3 - c_2) f(z_2), \end{aligned}$$

whose condition of compatibility is

$$(c_1 - c_3) f(z_0) + (c_2 - c_1) f(z_1) + (c_3 - c_2) f(z_2) = 0.$$

Computing the cohomology class of $\mu \wedge \mathrm{d}\mu$ we get

$$\int_{T^3} \mu \wedge \mathrm{d}\mu = (c_1 - c_3) f^2(z_0) + (c_2 - c_1) f^2(z_1) + (c_3 - c_2) f^2(z_2).$$

Taking $c_1 = 1$, $c_2 = -1$, $c_3 = 2$, $f(z_0) = 4$, $f(z_1) = 1$, $f(z_2) = 2$ and $1 \leqslant f(z) \leqslant 4$ for $0 \leqslant z \leqslant \frac{1}{3}$, $1 \leqslant f(z) \leqslant 2$ for $\frac{1}{3} \leqslant z \leqslant \frac{2}{3}$ and $2 \leqslant f(z) \leqslant 4$ for $\frac{2}{3} \leqslant z \leqslant 1$ we get $\int_{T^3} \mu \wedge \mathrm{d}\mu = -6$. By [1, 1.1] the $\mathscr{F}$-closed coframe $\{\mu, \mathrm{d}y\}$ determines a C^∞ action F with $\alpha_F \neq 0$. By [1, 1.3 and 3.1] F is not linearly equivalent to an action of the cylinder. Of course $\mathscr{F}$ has no first integrals and $\mathscr{F}$ is not the underlying foliation of an action of the cylinder $S^1 \times \mathbb{R}$.

2.4. If $T^p \to M \xrightarrow{\pi} T^2$ is a principal fiber bundle over T^2 we denote by $E = \{E_1, \ldots, E_p\}$ the frame of infinitesimal generators of the corresponding free action $E : T^p \times M \to M$ and by $\mathscr{E}$ the underlying foliation of E. The *restriction* to T^p of a nonsingular action $F : (T^p \times \mathbb{R}^S) \times M \to M$ is the action $F^p : T^p \times M \to M$ given by $F^p(v, x) = F(v, 0, x)$ for $v \in T^p$, $x \in M$. If $X = \{X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+S}\}$ is the frame of infinitesimal generators of F, then by [1, 2.1] there exists a Riemannian metric $\langle\, ,\rangle$ on M invariant under F^p such that X is orthonormal. Such a metric is said to be *adapted to* F. Thus if ν is the subbundle of TM normal to $T\mathscr{E}$, then it is invariant under F^p and $X_{p+1}, \ldots, X_{p+s}$ are sections of ν.

Next we show that Theorem 2.1 is false if $\mathscr{C}$ has C^r first integrals.

2.5. Example. Let $S^1 \to M \xrightarrow{\pi} T^2$ be a C^∞ circle bundle over T^2 and E_1 the infinitesimal generator of the corresponding free action $E : S^1 \times M \to M$. Let ε_1 be a connexion 1-form of π, i.e., $\varepsilon_1(E_1) = 1$ and ε_1 is invariant under E. Thus $\nu = \ker \varepsilon_1$ is a subbundle of TM normal to $T\mathscr{E}$ and invariant under E. Choose any nonvanishing C^∞ vector field $\bar{E}_2$ on T^2 with a C^∞ first integral $f > 0$. Let $\bar{\varepsilon}_2$ be a C^∞ 1-form on T^2 such that $\bar{\varepsilon}_2(\bar{E}_2) = 1$. Since f is nonconstant there exists a C^∞ function $g > 0$ such that

$$\int_{T^2} g \, \mathrm{d}f \wedge \bar{\varepsilon}_2 \neq 0. \tag{5}$$

Let E_2 be the lifting of $\bar{E}_2$ to a section of ν. Thus $\{E_1, E_2\}$ is a commuting 2-frame of M defining a C^∞ action of the cylinder $C : (S' \times \mathbb{R}) \times M \to M$. The 2-coframe $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$, $\varepsilon_2 = \pi^* \bar{\varepsilon}_2$ is adapted to C, and [1, 1] satisfies the relations

$$\mathrm{d}\varepsilon_1 \wedge \varepsilon_2 = \mathrm{d}\varepsilon_2 \wedge \varepsilon_2 = 0. \tag{6}$$

Let $\mathscr{C}$ be the underlying foliation of C. The 2-coframe $\xi = \{\xi_1, \xi_2\}$ where $\xi_1 = (f \circ \pi)\varepsilon_1$, $\xi_2 = (g \circ \pi)\varepsilon_2$ is $\mathscr{C}$-closed and defines by [1, 1.1] a C^∞ action F in $A^\infty(\mathbb{R}^2, M, \mathscr{C})$. It follows from (6) that

$$\xi_2 \wedge d\xi_2 = 0, \qquad d\xi_1 \wedge \xi_2 = (g \circ \pi)\, d(f \circ \pi)\varepsilon_1 \wedge \varepsilon_2.$$

Integration along the fibers of π, Fubini's theorem and (5) give

$$\begin{aligned} \int_M d\xi_1 \wedge \xi_2 &= \int_{T^2} \fint d\xi_1 \wedge \xi_2 \\ &= \int_{T^2} g\, df \wedge \bar{\varepsilon}_2 \neq 0. \end{aligned}$$

Thus $\det \alpha_F < 0$ and F is a hyperbolic action [1, 1.4]. By [1, 1.3 and 2.8] F is not linearly equivalent to an action of the cylinder $S^1 \times \mathbb{R}$.

We proved in [1, 1.12] that the set Ω of actions $F \in A^r(\mathbb{R}^2, M^3)$ with a nondegenerate characteristic mapping α_F is open in the C^1-topology. We now show that Ω is not necessarily dense, answering negatively [1, 1.13].

2.6. Example. It was proved in [6, 1.9] that there exists an open and dense set Σ of hyperbolic foliations $\mathscr{F}$ of T^2 such that the product foliations $S^1 \times \mathscr{F}$ on T^3 are C^1-stable. The foliations $S^1 \times \mathscr{F}$ are given by nonsingular C^∞ actions C of the cylinder $S^1 \times \mathbb{R}$ without first integrals. Now by [6, 1.9] and Corollary 2.2 there exists a neighborhood V of C in the C^1-topology in $A^\infty(\mathbb{R}^2, T^3)$ such that every action F in V has a vanishing characteristic mapping.

3. Concordance of actions of cylinders

To study the space of actions $A^r(\mathbb{R}^p, M)$ we considered two equivalence relations: *linear equivalence* and *conjugation* [2, 1 and 1.5]. We denote by $\boldsymbol{A}^r(\mathbb{R}^p, M)$ the set of all classes $\boldsymbol{F}$ of linear equivalence of nonsingular actions $F: \mathbb{R}^p \times M \to M$. Two actions $\boldsymbol{F}$ and $\boldsymbol{G}$ in $\boldsymbol{A}^r(\mathbb{R}^p, M)$ are said to be C^s conjugated if there exist $F \in \boldsymbol{F}$ and $G \in \boldsymbol{G}$ and a C^s diffeomorphism $\phi: M \to M$ such that

$$G(v, \phi(x)) = \phi(F(v, x)).$$

Two nonsingular actions of the cylinder with the same underlying foliation are not in general conjugate as the example below shows. For this reason we shall later consider another equivalence relation called *concordance.*

3.1. Example. Let $S^1 \to M \xrightarrow{\pi} T^2$ be a circle bundle over T^2, E_1 the infinitesimal generator of the corresponding free action of the circle and ε_1 a connexion form of π, i.e., ε_1 is a 1-form invariant under E_1 and such that $\varepsilon_1(E_1) = 1$. Thus $\nu = \ker \varepsilon_1$ is an E_1-invariant normal bundle. Consider the linear vector field $E_2^0 = \partial/\partial x - a\, \partial/\partial y$, $a \in \mathbb{R} - \mathbb{Q}$ on T^2 and let E_2 be its horizontal lifting. Since E_1 and E_2 commute they define an action F of $S^1 \times \mathbb{R}$ on M. Now if $f: T^2 \to \mathbb{R}$ is a C^∞ nonvanishing function, then $X_2 = (f \circ \pi) E_2$ also commutes with $X_1 = E_1$ and thus define an action $F^1: (S^1 \times \mathbb{R}) \times M \to M$. If F^0 and F^1 are conjugated by a diffeomorphism, then $\phi_* E_1 = E_1$ and the induced diffeomorphisms $\phi_0: T^2 \to T^2$ conjugates E_2^0 with cfE_2^0, c being a constant. We proved in [2, 2.9] that if a is Liouville number, then there are infinite many f for which this is impossible.

Let $\xi = (\xi_1, \ldots, \xi_p)$ be a p-coframe on M and $\Lambda(M)$ be the graded algebra of differentiable forms on M. We denote by $I(\xi)$ and $I(\mathrm{d}\xi)$ the ideals of $\Lambda(M)$ generated by $\xi_1, \ldots, \xi_p$ and $\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_p$, respectively. The notion of maximal class of 1-forms [4, VI 1.6] can be generalized to ξ as follows:

3.2. Definition. A p-coframe $\xi = (\xi_1, \ldots, \xi_p)$ on M has *maximal class* $2q+1$ if $I(\xi) \wedge I(\mathrm{d}\xi)^q \neq 0$ and $I(\mathrm{d}\xi)^{q+1} = 0$.

3.3. Example. If ξ is a p-coframe adapted to a nonsingular C^r, $r \geq 2$, action $F: \mathbb{R}^p \times M \to M$, then by [1, 1.1] the forms $\mathrm{d}\xi_j$ belong to the *annihilating ideal* $I(\mathcal{F})$ of the underlying foliation $\mathcal{F}$ of F and $I(\mathcal{F})^{q+1} = 0$ where q is the codimension of $\mathcal{F}$. Thus the maximal class of ξ is less or equal to $2q+1$.

3.4. Definition. Two C^r p-coframes ξ^0 and ξ^1 on M of maximal class less or equal to $2q+1$ are said to be *concordant* if there exists a p-tuple $\xi = (\xi_1, \ldots, \xi_p)$ of C^r 1-forms on $M \times \mathbb{R}$ such that

(i) $\xi(x, t) = (p^* \xi^0)(x, t)$ for $t \leq 0$ and all $x \in M$, $\xi(x, t) = (p^* \xi^1)(x, t)$ for $t \geq 1$ and all $x \in M$, $p: M \times \mathbb{R} \to M$ being the projection;

(ii) $I(\mathrm{d}\xi)^{q+1} = 0$.

If ξ^0 and ξ^1 are concordant, we write $\xi^0 \sim \xi^1$. Two p-coframes ξ^0 and ξ^1 adapted to a same nonsingular action $F: \mathbb{R}^p \times M \to M$ are concordant. For, if $w = \xi^1 - \xi^0$ and λ is any C^∞ function with $\lambda(t) = 0$, for $t \leq 0$, and $\lambda(t) = 1$ for $t \geq 1$, then $\xi = p^* \xi^0 + \lambda p^* w$ gives a concordance of ξ^0 with ξ^1. This is so because w_j, $\mathrm{d}\xi_j^0 \in I(\mathcal{F})$ and $I(\mathcal{F})^{q+1} = 0$, q being the codimension of $\mathcal{F}$.

Two nonsingular C^r, $r \geq 2$, actions F and G of $\mathbb{R}^p$ on M are *concordant* if their adapted coframes are concordant. It follows from the classical theorems of Stokes and de Rhan that two concordant actions have the same characteristic mappings. More generally we say that two actions $\boldsymbol{F}$ and $\boldsymbol{G}$ in $\boldsymbol{A}^r(\mathbb{R}^p, M)$ are concordant if there exist actions $F \in \boldsymbol{F}$ and $G \in \boldsymbol{G}$ which are concordant.

3.5. The notion of concordance is a generalization of the notion of cohomologous closed 1-forms. If a p-coframe ξ^0 has maximal class less or equal to $2q+1$, then $\xi^1 = \xi^0 + \mathrm{d}h$ also has maximal class less or equal to $2q+1$ and $\xi^0 \sim \xi^1$. In fact, if λ is a C^∞ function such that $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = 1$ for $t \geq 1$ and $H(x, t) = \lambda(t) h(x)$, then $\xi = p^* \xi^0 + \mathrm{d}H$, where $p: M \times \mathbb{R} \to M$ is the projection, is a concordance between ξ^0 and ξ^1.

The foliated cohomology, introduced by Reinhart in [5] appears naturally in the study of actions [2, 2.5]. If $\mathscr{F}$ is a foliation on M and ν is a normal bundle to $T\mathscr{F}$, then we say that a p-form $\mu \in \Lambda(M)$ is of type $(0, p)$ if $i_X\mu = 0$ for any section X of ν. We denote by $\Lambda^p(M, \mathscr{F})$ the space of p-forms of type $(0, p)$. We also consider the graded algebra

$$\Lambda(M, \mathscr{F}) = \bigoplus_{j=0}^{p} \Lambda^j(M, \mathscr{F}),$$

where p is the dimension of $\mathscr{F}$. $\Lambda(M, \mathscr{F})$ is a differential complex with respect to the *foliated differential*

$$\mathrm{d}_{\mathscr{F}} : \Lambda^j(M, \mathscr{F}) \to \Lambda^{j+1}(M, \mathscr{F})$$

defined as follows: If $\mu \in \Lambda^j(M, \mathscr{F})$, then $\mathrm{d}\mu$ can be written uniquely as $\mathrm{d}\mu = \alpha + \beta$ where α is of type $(0, j+1)$ and $\beta \in I(\mathscr{F})$. The operator $\mathrm{d}_{\mathscr{F}}$ is defined by $\mathrm{d}_{\mathscr{F}}\mu = \alpha$ [3]. A form $\mu \in \Lambda(M, \mathscr{F})$ is $\mathrm{d}_{\mathscr{F}}$*-closed* if $\mathrm{d}_{\mathscr{F}}\mu = 0$.

Two $\mathrm{d}_{\mathscr{F}}$-closed forms μ^0 and μ^1 in $\Lambda^1(M, \mathscr{F})$ are said to be *integrably homotopic* ($\mu^0 \simeq \mu^1$) if there exist a $\mathrm{d}_{\mathscr{F}\times\mathbb{R}}$-closed 1-form μ in $\Lambda^1(M \times \mathbb{R}, \mathscr{F} \times \mathbb{R})$ such that $\mu(x, t) = p^*\mu^0(x, t)$ for $t \leq 0$ and $\mu(x, t) = p^*\mu^1(x, t)$ for $t \geq 1$. We say that μ^0 and μ^1 are *cohomologous* and write $[\mu^0] = [\mu^1]$ if $\mu^1 = \mu^0 + \mathrm{d}_{\mathscr{F}}h$ for some differentiable functions $h : M \to \mathbb{R}$. We have the following:

3.6. Lemma. *Let μ^0 and μ^1 be $\mathrm{d}_{\mathscr{F}}$-closed forms of type $(0, 1)$. Then μ^0 and μ^1 are cohomologous if and only if they are integrably homotopic.*

Proof. Suppose $\mu^1 = \mu^0 + \mathrm{d}_{\mathscr{F}}h$ and let $\lambda : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = 1$ for $t \geq 1$. If $H : M \times \mathbb{R} \to \mathbb{R}$ is given by $H(x, t) = \lambda(t)h(x)$, then $\mu = p^*\mu^0 + \mathrm{d}_{\mathscr{F}\times\mathbb{R}}H$ is an integrable homotopy between μ^0 and μ^1. Conversely if $\mu \in \Lambda^1(M, \mathscr{F} \times \mathbb{R})$ is an integrable homotopy between μ^0 and μ^1, consider the homotopy operator $K : \Lambda(M \times \mathbb{R}, \mathscr{F} \times \mathbb{R}) \to (M, \mathscr{F})$ of degree -1, defined in [3]. From $\mathrm{d}_{\mathscr{F}} \circ K + K \circ \mathrm{d}_{\mathscr{F}\times\mathbb{R}} = J_1^* - J_0^*$ where $J_t : M \to M \times \mathbb{R}$, $J_t(x) = (x, t)$ for $t = 0$, 1 are the inclusions mappings, we get $\mathrm{d}_{\mathscr{F}}K\mu = \mu^1 - \mu^0$ proving that $[\mu^1] = [\mu^0]$. $\square$

We say that two ordered p-coframes μ^0 and μ^1 of $\mathrm{d}_{\mathscr{F}}$-closed forms of type $(0, 1)$ are *integrably homotopic* if $\mu_j^0 \simeq \mu_j^1$ for $1 \leq j \leq p$. Clearly $\mu^0 \simeq \mu^1$ implies $\mu^0 \sim \mu^1$. Thus integrable homotopy is an equivalence relation more restrictive than concordance.

3.7. Proposition. *$A^r(\mathbb{R}^m, T^m)$ has only one class of integrable homotopy.*

Proof. Let E be the canonical action of T^m on T^m and F be any nonsingular C^r action of $\mathbb{R}^m$ on T^m. If A is the matrix of the characteristic mapping α_F of F, then by [1, 1.3] the action $G(v, x) = F({}^tAv, x)$ has characteristic mapping $\alpha_G = I$. Thus if ξ is a coframe adapted to G, then $\xi = \mathrm{d}x + \mathrm{d}h$ where $\mathrm{d}x = \{\mathrm{d}x_1, \ldots, \mathrm{d}x_m\}$ is the canonical coframe of T^m and $h : T^m \to \mathbb{R}^m$ is a differentiable function. So, by Lemma 3.6 we have $\xi \simeq \mathrm{d}x$, proving the proposition. $\square$

Let $T^k \to M \xrightarrow{\pi} M$ be a principal fiber bundle. Thus we have a free action $T^k \times M \xrightarrow{\phi} M$ and M_0 is the space of the orbits of ϕ. A Riemannian metric $\langle\,,\rangle$ is *adapted* to π if the frame $E = \{E_1, \ldots, E_k\}$ of the infinitesimal generators of ϕ is orthonormal and $\langle\,,\rangle$ is invariant under ϕ. Associated to $\langle\,,\rangle$ we have the *connexion form* $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_k\}$ of π given by $\varepsilon_j(Y) = \langle E_j, Y\rangle$ where Y is any vector field on M. Let $\ker \varepsilon = \bigcap_{j=1}^{k} \ker \varepsilon_j$.

3.8. Lemma. *Let $T^k \to M \xrightarrow{\pi} M_0$ be a principal fiber bundle, $\langle\,,\rangle$ a Riemannian metric adapted to π and $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_k\}$ be its associated connexion form. If $\mathcal{F}_0$ is a foliation on M_0 without differentiable first integrals and the foliation $\mathcal{F} = \pi^*\mathcal{F}_0$ is such that $\ker \varepsilon \cap T\mathcal{F}$ is integrable, then each C^r, $r \geq 1$, $\mathrm{d}_{\mathcal{F}}$-closed form μ of type $(0, 1)$ can be written uniquely as*

$$\mu = a_1\varepsilon_1 + \cdots + a_k\varepsilon_k + \pi^*\eta + \mathrm{d}_{\mathcal{F}}h, \tag{7}$$

where $a_j \in \mathbb{R}$, $1 \leq j \leq k$, η is a $\mathrm{d}_{\mathcal{F}_0}$-closed form of type $(0, 1)$ on M_0 and $h : M \to \mathbb{R}$ is a C^r function.

Proof. If $\mathcal{E}$ is the foliation generated by E, then

$$T\mathcal{F} = T\mathcal{E} \oplus \ker \varepsilon \cap T\mathcal{F}.$$

Since the vector fields E_j commute and $[E_j, X]$ is a section of $\ker \varepsilon$ whenever X is a section of $\ker \varepsilon$, then $\mathrm{d}_{\mathcal{F}}\varepsilon_j = 0$ for $1 \leq j \leq k$ if and only if $\ker \varepsilon \cap T\mathcal{F}$ is integrable. Now suppose $k = 1$ and let μ be a $\mathrm{d}_{\mathcal{F}}$-closed C^r form of type $(0, 1)$ of M. Let $\langle\,,\rangle$ be a Riemannian metric on M adapted to the action of T^1 on M, ε_1 its associated connexion from and $\nu = \ker \varepsilon_1$.

The metric $\langle\,,\rangle$ induces a metric $\langle\,,\rangle_0$ on M_0 by $\langle u, v\rangle_0 = \langle \bar{u}, \bar{v}\rangle$, $u, v \in T_{\pi(x)}M_0$ where $\bar{u}$, $\bar{v} \in \nu_x$ are horizontal liftings of u and v. Cover M_0 by a finite number of open sets u_k, $1 \leq k \leq n$, in which are defined $(m-1)$-frames $\mathcal{B}_k = \{X_1^0, \ldots, X_p^0, Y_1^0, \ldots, Y_q^0\}$ such that $[X_i^0, X_j^0] = 0$, $1 \leq i, j \leq p$ and $\langle X_i^0, Y_l^0\rangle = 0$, and X_i^0 are tangent to $\mathcal{F}_0$, for $1 \leq i \leq p$, $1 \leq l \leq q$. Let $\{\xi_1^0, \ldots, \xi_p^0, \eta_1^0, \ldots, \eta_q^0\}$ be the dual coframe of $\mathcal{B}_k$. Thus in each open set $\pi^{-1}(u_k)$ is defined an m-coframe $\{\varepsilon, \xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_q\}$ where $\xi_j = \pi^*\xi_j^0$ and $\eta_l = \pi^*\eta_l^0$, $1 \leq j \leq p$, $1 \leq l \leq q$. Let $\mathcal{P} = \{\phi_k\}$ be a C^∞ partition of unity subordinated to $\{u_k\}$. Since $\mathrm{d}_{\mathcal{F}_0} \oint \mu = \oint \mathrm{d}_{\mathcal{F}}\mu = 0$ it follows that $\oint \mu$ is constant on the leaves of $\mathcal{F}_0$ and thus by hypothesis $a_1 = \oint \mu$ is constant. So $\mu_0 = \mu - a_1\varepsilon$ is $\mathrm{d}_{\mathcal{F}}$-closed, $\oint \mu_0 = 0$ and

$$\mu_0 = f\varepsilon_1 + \mu_1,$$

where $f = i_{E_1}\mu_0$ and $i_{E_1}\mu_1 = 0$. Thus μ_0 can be written in each $\pi^{-1}(u_k)$ as

$$\mu_0 = f\varepsilon_1 + \sum_{j=1}^{p} g_j\xi_j. \tag{8}$$

From $\oint \mu_0 = \oint f\varepsilon = 0$ it can be seen that there exists a differentiable function $h: M \to \mathbb{R}$ such that

$$f = E_1 h. \tag{9}$$

Now from (8) and (9) we get

$$\mu_0 - \mathrm{d}_{\mathscr{F}} h = \sum_{j=1}^{p} (g_j - X_j h)\xi_j \tag{10}$$

on $\pi^{-1}(u_k)$ where X_j are horizontal liftings of X_j^0, $1 \leq j \leq p$.

Since E_1 commutes with each X_j, $1 \leq j \leq p$, and $f = E_1 h$, then $\mathrm{d}_{\mathscr{F}}\mu_0 = 0$ implies

$$0 = E_1 g_j - X_j f = E_1(g_j - X_j h), \tag{11}$$

showing that $g_j - X_j h$ are basic functions. Thus

$$\mu_0 - \mathrm{d}_{\mathscr{F}} h = \pi^* \theta_k \quad \text{on } \pi^{-1}(u_k), \tag{12}$$

where θ_k are 1-forms on u_k. If $\eta = \sum_k \phi_k \theta_k$, then $\mu_0 - \mathrm{d}_{\mathscr{F}} h = \pi^* \eta$ and η is a $\mathrm{d}_{\mathscr{F}}$-closed form of type $(0, 1)$. So

$$\mu = a_1 \varepsilon_1 + \pi^* \eta + \mathrm{d}_{\mathscr{F}} h. \tag{13}$$

Now, suppose as an induction assumption that the lemma is true for $k-1$. Let M_1 be the manifold consisting of the orbits of E_1. We have two principal fiber bundles $T^1 \to M \xrightarrow{\pi} M_1$ and $T^{k-1} \to M_1 \xrightarrow{\pi_0} M_0$. Let $\mathscr{F}'$ be the foliation given by $\mathscr{F}^1 = \pi_0^* \mathscr{F}_0$. Since $\mathrm{d}_{\mathscr{F}}\varepsilon_1 = 0$ from (13) we get

$$\mu = a_1 \varepsilon_1 + \pi_1^* \eta_1 + \mathrm{d}_{\mathscr{F}'} h_1, \tag{14}$$

where η_1 is a $\mathrm{d}_{\mathscr{F}'}$-closed form of type $(0, 1)$ on M and h_1 is a differentiable function. Let $\varepsilon' = \{\varepsilon_2', \ldots, \varepsilon_k'\}$ be the connexion form of π_0 such that $\pi_1^* \varepsilon_j' = \varepsilon_j$ for $2 \leq j \leq k$. Now as ε_j are $\mathrm{d}_{\mathscr{F}}$-closed forms, then ε_j' are $\mathrm{d}_{\mathscr{F}'}$-closed forms for $2 \leq j \leq k$. Thus by the induction assumption

$$\eta_1 = a_2 \varepsilon_2' + \cdots + a_k \varepsilon_k' + \pi_0^* \eta + \mathrm{d}_{\mathscr{F}'} h_0. \tag{15}$$

From (14) and (15) we get (7), proving the lemma. □

Denote by C_k the restriction to T^k of an action C of the cylinder $T^k \times \mathbb{R}^l$ on M. We say that a Riemannian metric $\langle\,,\rangle$ is *adapted* to C if it is invariant under C_k and the frame of the infinitesimal generators of C is an orthonormal set.

3.9. Theorem. *Let $\mathscr{C}$ be a foliation on a closed orientable connected m-manifold M given by a C^r, $r \geq 2$, nonsingular action C of the cylinder $T^{m-2} \times \mathbb{R}$. If $\mathscr{C}$ has no C^r first integrals, then every action $F \in A^r(\mathbb{R}^{m-1}, M, \mathscr{C})$ is concordant to C.*

Proof. By [1, 2.4] we may assume that, up to a linear equivalence, C_{m-2} is a free action and by [1, 2.7] we have a C^r action $C_0: \mathbb{R} \times M_0 \to M_0$ where M_0 is the space of C_{m-2} orbits. M_0 is a manifold diffeomorphic to T^2 and the projection $\pi: M \to M_0$ is a submersion. The actions C and C_0 are related by

$$C_0(v, \pi(x)) = \pi C((u, v), x),$$

where $(u, v) \in T^{m-2} \times \mathbb{R}$ and $x \in M$. Choose a Riemannian metric $\langle\, ,\rangle$ adapted to C. Now let $F \in A^r(\mathbb{R}^{m-1}, M, \mathscr{C})$ and $\mu = \{\mu_1, \ldots, \mu_{m-1}\}$ be the coframe of type (0, 1) adapted to F. By Lemma 3.8 each 1-form μ_i can be written uniquely as

$$\mu_i = a_{i1}\varepsilon_1 + \cdots + a_{im-2}\varepsilon_{m-2} + (f_i \circ \pi)\xi + \mathrm{d}_{\mathscr{F}} h_i, \tag{16}$$

where $a_{ij} \in \mathbb{R}$, $f_i: M_0 \to \mathbb{R}$ and $h_i: M \to \mathbb{R}$ are C^r functions and $\xi = \pi^*\xi_0$ where ξ_0 is a 1-form of type (0, 1) adapted to C_0. If w_0 is a 1-form on M_0 defining $\mathscr{F}_0$, then $w = \pi^* w_0$ defines $\mathscr{F}$ and $\mu_1 \wedge \cdots \wedge \mu_{m-1} \wedge w$ is a volume element. Thus, by Fubini's theorem,

$$\int_M \mu \wedge \cdots \wedge \mu_{m-1} \wedge w = \int_{M_0} (\textstyle\oint \mu_1 \wedge \cdots \wedge \mu_{m-1}) \wedge w_0 \neq 0. \tag{17}$$

It follows from (16) and (17) that the $(m-1) \times (m-2)$ matrix $A = (a_{ij})$ has rank $m-2$. Thus after a sequence of elementary row operations we may assume, up to a linear equivalence, that

$$\mu_j = \varepsilon_j + (f_j \circ \pi)\xi + \mathrm{d}_{\mathscr{F}} h_j, \qquad 1 \leq j \leq m-2, \tag{18}$$

and

$$\mu_{m-1} = (f_{m-1} \circ \pi)\xi + \mathrm{d}_{\mathscr{F}} h_{m-1}.$$

By Lemma 3.6 we have integrable homotopies

$$\mu_j \simeq \mu_j^0 = \varepsilon_j + (f_j \circ \pi)\xi, \quad 1 \leq j \leq m-2, \tag{19}$$

and

$$\mu_{m-1} \simeq \mu_{m-1}^0 = (f_{m-1} \circ \pi)\xi.$$

We now construct a concordance between μ^0 and $c = \{\varepsilon_1, \ldots, \varepsilon_{m-2}, \xi\}$, which is a coframe adapted to C.

Let $\lambda: \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\lambda(t) = 1$ for $t \leq 0$ and $\lambda(t) = 0$ for $t \geq 1$. Let ζ be the $(m-1)$-tuple of 1-forms given by

$$\zeta_j = p^*\varepsilon_j + \lambda(f_j \circ \pi)p^*\xi, \quad 1 \leq j \leq m-2,$$

and

$$\zeta_{m-1} = [(1-\lambda) + \lambda(f_{m-1} \circ \pi)]p^*\xi.$$

Since $\xi = \pi^*\xi_0$ where ξ_0 is a 1-form on the two-dimension manifold M_0 and $\mathrm{d}\varepsilon_j \wedge \xi = 0$, $1 \leq j \leq m-2$, it follows that $\mathrm{d}\zeta_i \wedge \mathrm{d}\zeta_j = 0$, $1 \leq i, j \leq m-1$. Thus ζ is a concordance between μ^0 and c. From transitivity of concordance $\mu \sim c$, proving the theorem. $\square$

Next we show that any two C^r, $r \geq 2$, free actions of T^{m-2} on M are conjugate by a C^r diffeomorphism. We first consider the manifolds M_n^m. To each $n \in \mathbb{Z}^{m-2}$ we associate the matrix $A_n \in \mathrm{SL}(m-1, \mathbb{R})$

$$A_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & n_1 \\ 0 & 1 & \cdots & 0 & n_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & n_{m-2} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

M_n^m is the m-manifold obtained from $T^{m-1} \times [0, 1]$ by identifying $(x, 1)$ with $(A_n(x), 0)$. Let $\{\partial/\partial x_1, \ldots, \partial/\partial x_{m-2}, \partial/\partial x_{m-1}\}$ be the canonical frame of T^{m-1}. Since $(A_n)_* \partial/\partial x_i = \partial/\partial x_i$, $1 \leq i \leq m-2$, each $\partial/\partial x_i$ defines a vector field on M_n^m also denoted by $\partial/\partial x_i$, $1 \leq i \leq m-2$. Thus $\partial/\partial x = \{\partial/\partial x_1, \ldots, \partial/\partial x_{m-2}\}$ is a commutative $(m-2)$-frame generating the *canonical action* E of T^{m-2} on M_n^m.

3.10. Lemma. *Let $\phi : T^n \to T^n$ be a diffeomorphism such that $\phi_* \, \partial/\partial x_i = \partial/\partial x_i$, $1 \leq i \leq k$, $n-3 \leq k \leq n$. Then ϕ is isotopic to a diffeomorphism A induced by a matrix $A \in \mathrm{SL}(n, \mathbb{Z})$ whose columns are the canonical vectors e_j, $1 \leq j \leq k$.*

Proof. The lifting of ϕ to $\mathbb{R}^n$ is $\phi = A + \beta$ where $A \in \mathrm{SL}(n, \mathbb{Z})$ and $\beta = (\beta_1, \ldots, \beta_n)$ is $\mathbb{Z}^n$-periodic. The hypothesis implies that the k first columns of A are $A^j = {}^t e_j$ and the β_j depend only on the last $n-k$ variables $x_{k+1}, \ldots, x_n$. Thus ϕ restricts to $T_{k+1}^1 \times \cdots \times T_n^1$ as a diffeomorphism and

$$\phi_t = A + (t\beta_1, \ldots, t\beta_k, \beta_{k+1}, \ldots, \beta_n)$$

gives an isotopy between $\phi = A + \beta$ and $\phi_0 = A + (0, \ldots, 0, \beta_{k+1}, \ldots, \beta_n)$. By the theorem on isotopies for dimension $n \leq 3$, ϕ_0 is isotopic to A, proving the lemma. □

3.11. Theorem. *Let $T^{m-2} \to M \xrightarrow{\pi} T^2$ be a principal C^r, $r \geq 2$, fiber bundle and $E = \{E_1, \ldots, E_{m-2}\}$ be the frame adapted to the corresponding free action of T^{m-2} on M. Then there exist a vector $n \in \mathbb{Z}^{m-2}$ and a diffeomorphism $\chi : M_n^m \to M$ such that $\chi_* \partial/\partial x_j = E_j$, $1 \leq j \leq m-2$.*

Proof. Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_{m-2}\}$ be a connexion form on π and $\nu = \ker \varepsilon$ its invariant normal bundle. Consider the projection $p : T^2 \to T^2$ on the first factor of $T^2 = T^1 \times T^1$ and let $\pi_1 = p \circ \pi$. Denote by $\{\partial/\partial z_1, \partial/\partial z_2\}$ and $\{\partial/\partial z\}$ the canonical frames of T^2 and T^1, respectively. Fix a fiber $F_z = \pi_1^{-1}(z)$, $z \in T^1$ and denote by $E^z = \{E_1^z, \ldots, E_{m-2}^z\}$ the restriction of E to F_z. Let X and Z be the horizontal liftings of $\partial/\partial z_2$ and $\partial/\partial z$ to M, respectively. As X is tangent to F_z and commutes with each E_j^z, then F_z is diffeomorphic to T^{m-1}. The C^r mapping

$$\zeta : F_z \times [0, 1] \to M$$

given by $\zeta(y, t)=\zeta_t(y)$, ζ_t being the flow of Z induces a diffeomorphism

$$\zeta:\frac{F_z\times[0,1]}{\zeta_1}\to M, \tag{20}$$

such that $\zeta_*\partial/\partial t=Z$ and $\zeta_* E_j^z=E_j$, $1\leq j\leq m-2$. Choose a C^r diffeomorphism $\phi: T^{m-1}\to F_z$ such that $\phi_*\partial/\partial x_j=E_j^z$, $1\leq j\leq m-2$. Let

$$\Phi:\frac{T^{m-1}\times[0,1]}{\phi^{-1}\circ\zeta_1\circ\phi}\to\frac{F_z\times[0,1]}{\zeta_1} \tag{21}$$

be the C^r diffeomorphism induced by $\phi\times\mathrm{id}: T^{m-1}\times[0,1]\to F_z\times[0,1]$.

Let $\psi=A+\beta$ be the lifting of $\phi^{-1}\zeta\phi$ to $\mathbb{R}^m$. Since $\psi_*\partial/\partial x_j=\partial/\partial x_j$ for $1\leq j\leq m-2$, it follows from Lemma 3.10 that $A=A_n$ for some $n\in\mathbb{Z}^{m-2}$ and there exists an isotopy ψ_t between A_n and ψ such that $(\psi_t)_*\partial/\partial x_j=\partial/\partial x_j$, $1\leq j\leq m-2$. Thus the mapping $\psi(x,t)=(\psi_{1-t}(x),t)$ induces a diffeomorphism

$$\psi:\frac{T^{m-1}\times[0,1]}{A_n}\to\frac{T^{m-1}\times[0,1]}{\phi^{-1}\circ\zeta_1\circ\phi} \tag{22}$$

such that $\psi_*\partial/\partial x_j=\partial/\partial x_j$, $1\leq j\leq m-2$.

It follows from (20)-(22) that there exists a C^r diffeomorphism $\chi: M_n^m\to M$ which conjugates $\partial/\partial x_j$ with E_j, $1\leq j\leq m-2$, proving the theorem. □

Let $F:(T^{m-2}\times\mathbb{R})\times M\to M$ be a C^r, $r\geq 2$, nonsingular action such that the restriction F^{m-2} of F to T^{m-2} is free. Let $\{E_1,\ldots,E_{m-2},X\}$ be the frame of the infinitesimal generators of F and $T^{m-2}\to M\xrightarrow{\pi} M_0$ the principal fiber bundle *associated* to F^{m-2}. Let $F_0:\mathbb{R}\times M_0\to M_0$ be the quotient action of F, i.e., $F_0(v,\pi(x))=F((0,v),x)$ and X_0 its infinitesimal generator. Fix a connexion $\varepsilon=\{\varepsilon_1,\ldots,\varepsilon_{m-2}\}$ of π. The vector field X can be written as $X=X^{\mathrm{v}}+X^{\mathrm{h}}$ where X^{v} is vertical and X^{h} is horizontal. Since X commutes with each E_j, $1\leq j\leq m-2$, it follows that $[E_j,X^{\mathrm{v}}]=[E_j,X^{\mathrm{h}}]=0$ for $1\leq j\leq m-2$. Thus $X^{\mathrm{v}}=\sum_{j=1}^{m-2}a_jE_j$ where $a_j=\varepsilon_j^0(X)$ and X^{h} is the horizontal lifting of X_0. We denote by $F^{\mathrm{h}}:(T^{m-2}\times\mathbb{R})\times M\to M$ the C^r nonsingular action generated by $\{E_1,\ldots,E_{m-2},X^{\mathrm{h}}\}$. F^{h} will be refered to as the *horizontal projection* of F.

Now we show that any two nonsingular actions of the cylinder $T^{m-2}\times\mathbb{R}$ on M are (up to a conjugation by a diffeomorphism) concordant.

Let E be the canonical action of T^{m-2} on M_n^m and $\mathscr{E}$ be its underlying foliation. We denote by $A^r(T^{m-2}\times\mathbb{R}, M_n^m,\mathscr{E})$ the set of all C^r, $r\geq 2$, nonsingular actions F of $T^{m-2}\times\mathbb{R}$ on M_n^m whose restriction F^{m-2} to T^{m-2} is free and with the same underlying foliation $\mathscr{E}$. $A^r(T^{m-2}\times\mathbb{R}, M)$ will denote the set of all classes of linear equivalence of C^r nonsingular actions of $T^{m-2}\times\mathbb{R}$ on M.

3.12. Theorem. *Let M be a closed connected orientable m-manifold. Then*

(i) *Each action F in $A^r(T^{m-2}\times\mathbb{R}, M)$ is conjugate by a C^r diffeomorphism to an action F_0 in $A^r(T^{m-2}\times\mathbb{R}, M_n^m,\mathscr{E})$ for some $n\in\mathbb{Z}^{m-2}$.*

(ii) *$A^r(T^{m-2}\times\mathbb{R}, M_n^m,\mathscr{E})$ has only one concordance class, for each $n\in\mathbb{Z}^{m-2}$.*

Proof. (i) By [1, 2.4] there exists $F \in \boldsymbol{F}$ whose restriction F^{m-2} to T^{m-2} is free. Thus the result follows from Theorem 3.11.

(ii) Choose a connexion $\varepsilon^0 = \{\varepsilon_1^0, \ldots, \varepsilon_{m-2}^0\}$ on the canonical fiber bundle $T^{m-2} \to M_n^m \to T^2$. First we show that each $F \in A^r(T^{m-2} \times \mathbb{R}, M_n^m, \mathscr{E})$ is concordant to its horizontal projection F^{h}. Let ξ_0 be a c^r 1-form on T^2 such that $\xi_0(X_0) = 1$ and $\xi^{\mathrm{h}} = \pi^* \xi_0$. The $(m-1)$-coframe $\{\varepsilon^0, \xi^{\mathrm{h}}\}$ is adapted to F^{h} and $\{\varepsilon_1^0 - a_1 \xi^{\mathrm{h}}, \ldots, \varepsilon_{m-2} - a_{m-2}\xi^{\mathrm{h}}, \xi^{\mathrm{h}}\}$ where $a_j = \varepsilon_j^0(X)$, $1 \leq j \leq m-2$, is adapted to F. Consider the 1-forms ε_j on $M_n^m \times \mathbb{R}$ given by $\varepsilon_j = p^* \varepsilon_j^0 - \lambda(t)(a_j \circ p) p^* \xi^{\mathrm{h}}$, $1 \leq j \leq m-2$, where $p: M_n^m \times \mathbb{R} \to M_n^m$ is the projection and $\lambda: \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\lambda(t) = 0$ for $t \leq 0$, $\lambda(t) = 1$ for $t \geq 1$. The $(m-1)$-coframe $\{\varepsilon_1, \ldots, \varepsilon_{m-2}, \xi\}$ on $M_n^m \times \mathbb{R}$ where $\xi = p^* \xi^{\mathrm{h}}$ gives a concordance of F^{h} with F. To see this it is sufficient to observe that $\mathrm{d}\varepsilon_j^0 \wedge \xi^{\mathrm{h}} = 0$, $1 \leq j \leq m-2$, and $\xi^{\mathrm{h}} \wedge \mathrm{d}\xi^{\mathrm{h}} = \pi^*(\xi_0 \wedge \mathrm{d}\xi_0) = 0$ since $\xi^{\mathrm{h}} = \pi^* \xi_0$ and ξ_0 is a 1-form on T^2.

Now, we show that if $F, G \in A^r(T^{m-2} \times \mathbb{R}, M_n^m, \mathscr{E})$, then F^{h} is concordant to G^{h}. Choose a 1-form η_0 on T^2 such that $\{\varepsilon^0, \eta^{\mathrm{h}}\}$ where $\eta^{\mathrm{h}} = \pi^* \eta_0$ is a coframe adapted to G^{h}. Let ξ be the 1-form on $M_n^m \times \mathbb{R}$ given by $\xi = (1-\lambda) p^* \xi^{\mathrm{h}} + \lambda p^* \eta^{\mathrm{h}}$. Since $\mathrm{d}\varepsilon_j^0 \wedge \xi^{\mathrm{h}} = \mathrm{d}\varepsilon_j^0 \wedge \eta^{\mathrm{h}} = 0$, $1 \leq j \leq m-2$, and $\xi^{\mathrm{h}} = \pi^* \xi_0$, $\eta^{\mathrm{h}} = \pi^* \eta_0$ where ξ_0 and η_0 are 1-forms on T^2, then $\{p^* \varepsilon^0, \xi\}$ gives a concordance of F^{h} with G^{h}. □

3.13. Remark. Theorems 3.9 and 3.12 are not true for nonsingular actions of $T^k \times \mathbb{R}^l$ if $0 < k < m-2$. To see this consider on T^4 the canonical free action generated by $E = \partial/\partial x_4$. We have the canonical bundle $T^1 \to T^4 \xrightarrow{\pi} T^3$ with connexion form $\varepsilon = \mathrm{d}x_4$. Any nonsingular action $F_0: \mathbb{R}^2 \times T^3 \to T^3$ lifts to an action F of $T^1 \times \mathbb{R}^2$ on T^4 which is π-related to F_0, i.e.,

$$F_0(v, \pi(x)) = \pi F((u, v), x), \quad u \in T^1, \ v \in \mathbb{R}^2.$$

By [1, 2.6], the characteristic mappings of F and F_0 are related by

$$\alpha_F((0, v_1), (0, v_2)) = \pi^* \alpha_F(v_1, v_2),$$

where $v_1, v_2 \in \mathbb{R}^2$. Thus by [1, 1.5], $A^r(T^1 \times \mathbb{R}^2, T^4, \mathscr{E})$ has many concordance classes.

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